# Chapter 1: Analytic Geometry

## 1.1 Lines

- We use **delta (Δ)** to denote the change in a variable: $\Delta x = x_2 - x_1$
- The **slope between two points**, $A(x_1, y_1)$ and $B(x_2, y_2)$ is $m = \Delta y/\Delta x$
- **Standard equation of a line**: $y = mx + b$
- **Point slope form**: $(y - y_1) = m(x - x_1)$
    - Can be used to find the y-intercept by subbing in either point and letting $x = 0$, or $y = 0$ for the x-intercept
        - The logic is that $m$ made with either $A$ or $B$ and an intercept should be the same as $m$ between $A$ and $B$

## 1.2 Distance Between Two Points; Circles

- Between two points in 2D space a triangle can be formed from $\Delta x$ and $\Delta y$. Thus the **positive distance** between them is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$
- The distance between point $A$ to the origin is $r = \sqrt{(x - 0)^2 + (x - 0)^2}$, which becomes $r^2 = x^2 + y^2$; the *equation of a circle* centred at the origin
    - The special case $r = 1$ is called the **unit circle**
- The **equation of a circle** centred to $C(h, k)$ is $r^2 = (x - h)^2 + (x - k)^2$

***EXAMPLE 1.2.1***

*Graph the circle $x^2 - 2x + y^2 + 4y - 11 = 0$.*

We first have to complete the squares:

$$(x^2 - 2x + (\tfrac{2}{2})^2 - (\tfrac{2}{2})^2) + (y^2 + 4y + (\tfrac{4}{2})^2 - (\tfrac{4}{2})^2) - 11 = 0$$

$$(x^2 - 2x + 1 - 1) + (y^2 + 4y + 4 - 4) - 11 = 0$$

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) - 11 - 1 - 4 = 0$$

$$(x - 1)^2 + (y + 2)^2 = 16$$

∴ The circle has a radius of $\sqrt{16} = 4$ and centre of $C(1, -2)$

## 1.3 Functions

- Given a value of $x$, a **function** must give <u>at most one value</u> of $y$
    - Only multiple values are an issue; if $x$ has no $y$ for a value, no issue
- **Domain**: the *set* of $x$-values we're allowed to evaluate for the function
    - E.g. $x \geq 0$, $\{x \in R \mid x \geq 0\}$, or $[0, \infty)$

- Limited normally due to undefined behaviour, but "story problems" may imply additional non-pure math restrictions
- In "story problems" $x$ often plays the role of the **independent variable** and $y$ the **dependent variable**

***EXAMPLE 1.3.1***

An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side $x$ from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume $V$ of the box as a function of $x$, and find the domain of this function.

$V(x) = x * (a - 2x) * (b - 2x)$

∴ We know that the cut cannot be negative, nor can it be half of one the cardboard pieces measurements, thus $\{x \in R \mid 0 < x < \frac{1}{2} min(a, b)\}$.

***EXAMPLE 1.3.3***

Find the domain of $y = f(x) = \frac{1}{\sqrt{4x - x^2}}$

Since the denominator cannot be 0 and you cannot take the square of a negative, we need to find where $4x - x^2 > 0$:

$4x - x^2 > 0$

$x(4 - x) > 0$, thus $x > 0$ and $x > 4$

∴ $\{x \in R \mid x > 0\}$

## 1.4 Shifts and Dilations

- They cover the standard vertical and horizontal shift of parabolas
- You can also move a circle's centre: $r^2 = (x - h)^2 + (x - k)^2$
- You can also do horizontal and vertical dilation by multiplying/dividing $y$ and $x$ respectively by a value
    - A coefficient above 0 compresses and a below stretches
- With this we can create an **ellipse equation**: $(\frac{x}{a} - h)^2 + (\frac{y}{b} - k)^2 = 1$

# Chapter 2: Instantaneous Rate of Change: The Derivative

## 2.1 The Slope of a Function

- **Chord**: a line between two points on a circle
- The derivative of $f(x)$ is found using the **difference quotient**: $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

***Finding the Derivative of a Square Root Function***

*Find the Derivative of* $y = f(x) = \sqrt{625 - x^2}$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} * \frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}$$

$$= \lim_{\Delta x \to 0} \frac{(625-(x+\Delta x)^2)-(625-x^2)}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} = \lim_{\Delta x \to 0} \frac{-(x^2+2x\Delta x+\Delta x^2)+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x-\Delta x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} = \lim_{\Delta x \to 0} \frac{-2x-\Delta x}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}$$

$$= \frac{-2x}{2\sqrt{625-x^2}} = \frac{-x}{\sqrt{625-x^2}}$$

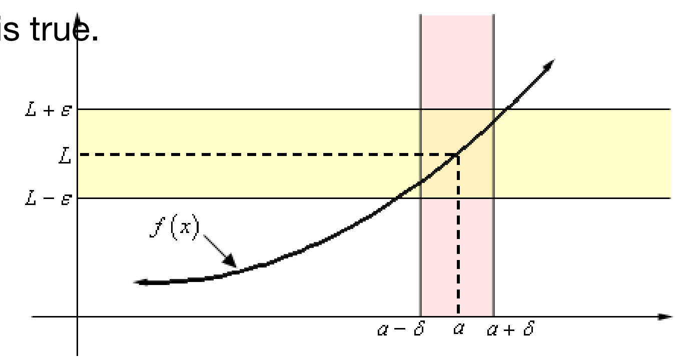
### ***Finding the Derivative of a Parabolic Function*** (2.2 An Example)

Suppose that the height of a dropped object can be measured with an equation of the form: $h(t) = h_0 - kt^2$. If $h_0 = 100\, metres$ and $k = 4.9\, metres/second$, find the derivative $h'(t)$.

$$h'(t) = \lim_{\Delta t \to 0} \frac{(100-4.9t^2) - (100-4.9(t+\Delta t)^2)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-4.9\Delta t^2-9.8t\Delta t)}{\Delta t} = -9.8t$$

## 2.3 Limits

- **Definition of a limit**: $\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
    - What this is really saying is given our error bars $\delta$ and $\varepsilon$, which must be greater than zero, that $x$ must be within the error bars for $\delta$ and the value at $x$, $f(x)$, must be within the error bars for $\varepsilon$. If we can find a relation between $\varepsilon$ and that makes this true close to both being zero, the limit is true.




### ***EXAMPLE 2.3.3***

*Prove:* $\lim_{x \to 2} (x + 4) = 6$

If $\delta > 0$ and $\varepsilon > 0$, and if $0 < |x - 2| < \delta$ then $|x + 4 - 6| < \varepsilon$

This example is simple as $|x + 4 - 6| < \varepsilon \Rightarrow |x - 2| < \varepsilon$, thus $\varepsilon = \delta$

### ***Proving the Limit of a Linear Function using the Definition of a Limit***

*Prove:* $\lim_{x \to 4} (2x + 3) = 11$. *That is, show that if* $0 < |x - a| < \delta$ *then* $|f(x) - L| < \varepsilon$

If $|x - 4| < \delta$ then $|(2x + 3) - 11| < \varepsilon \Rightarrow$

$|2x - 8| < \varepsilon \Rightarrow |2(x - 4)| < \varepsilon \Rightarrow |(x - 4)| < \varepsilon/2$

Let $\delta = \varepsilon/2$,

Now just reverse steps to prove the limit (some steps removed here)

If $|(x - 4)| < \delta \quad \Rightarrow \quad |(x - 4)| < \varepsilon/2 \quad \Rightarrow \quad |2x - 11| < \varepsilon$

Then $|f(x) - L| < \varepsilon$

***EXAMPLE 2.3.4***

*Prove:* $\lim_{x \to 2} (x^2) = 4$

If $\delta > 0$ and $\varepsilon > 0$, and if $0 < |x - 2| < \delta$ then $|x^2 - 4| < \varepsilon \Rightarrow$

$|(x - 2)(x + 2)| < \varepsilon \quad \Rightarrow$

***[Proving the Limit of a Quadratic Function using the Definition of a Limit]()*** ***(1)***

*Prove:* $\lim_{x \to 2} (x^2 - 4x + 5) = 1.$

Let $\varepsilon > 0$ be given; find $\delta > 0$:

$|x^2 - 4x + 5 - 1| < \varepsilon$ if $|x - 2| < \delta$

$|(x - 2)(x - 2)| < \varepsilon \quad \Rightarrow \quad |(x - 2)^2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \sqrt{\varepsilon}$

Let $\delta = \sqrt{\varepsilon}$ (reversing steps left out for brevity, but must be done in actual work):

$|x - 2| < \delta \quad \Rightarrow \quad |x - 2| < \sqrt{\varepsilon} \quad \Rightarrow \quad |x^2 - 4x + 5 - 1| < \varepsilon$

***[Proving the Limit of a Quadratic Function using the Definition of a Limit]()*** ***(2)***

- Stefan said this will not be on exams or quizzes.

*Prove:* $\lim_{x \to 1} (x^2 + 5x + 6) = 1.$

- $\forall$ means "for all", $\exists$ means "there exists", s.t. Means "such that"

$\forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t. $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

Proof:

$|x^2 + 5x + 6 - 12| < \varepsilon$ if $0 < |x - 1| < \delta$

$|x^2 + 5x - 6| < \varepsilon \quad \Rightarrow \quad |(x + 6)(x - 1)| < \varepsilon \Rightarrow$

$|x + 6| * |x - 1| < \varepsilon$

Problem: we control $|x - 1|$ through $\delta$, but not $|x + 6|$.

Say $|x - 1| < 1$ (1 is arbitrary but small). We need to get this to be $x + 6$

$|x - 1| < 1 \quad \Rightarrow \quad 0 < x < 2 \quad \Rightarrow \quad 6 < x + 6 < 8 \Rightarrow$

$|x + 6| < 8$

So we know $|x + 6|$ must be less than 8 if within 1 unit of the limit

Proof Continued:

$8|x - 1| < \varepsilon \quad \Rightarrow \quad |x - 1| < \varepsilon/8$

Let $\delta = min(1, \varepsilon/8)$

***General "Practical" Rules for Calculating the Limit***

Given $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

1. **Sum/Difference Rule**: $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$
2. **Constant Rule**: $\lim_{x \to a} k = k$, where $k$ is a constant
3. **Multiplication by Constant**: $\lim_{x \to a} [f(x) * k] = \lim_{x \to a} f(x) * k = L * k$
4. **Product Rule**: $\lim_{x \to a} [f(x) * g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x) = L * M$
5. **Quotient Rule**: $\lim_{x \to a} [f(x) \div g(x)] = \lim_{x \to a} f(x) \div \lim_{x \to a} g(x) = L \div M$
6. **Power Rule**: $\lim_{x \to a} [f(x)]^k = [\lim_{x \to a} f(x)]^k$
7. **Base Rule**: for any real number $b > 0$, $\lim_{x \to a} b^{f(x)} = b^{\lim_{x \to a} f(x)}$
8. **Polynomial Rule**: $\lim_{x \to a} p(x) = p(a)$, where $p(x)$ is a polynomial
9. **Log Rule**: $\lim_{x \to a} [log_b f(x)] = log_b [\lim_{x \to a} f(x)]$ if $0 < b < 1$ or $b > 1$ and $f(x) > 0$
10. **Root Rule**: $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, provided that $a \geq 0$ if $n$ is even
11. **Squeeze Theorem**: if $f(x) \leq h(x) \leq g(x)$, then $\lim_{x \to a} h(x) = L$

Given $\lim_{x \to L} f(x) = f(L)$ and $\lim_{x \to a} g(x) = L$

12. **Composition Rule**: $\lim_{x \to a} f(g(x) = f(L)$

***EXAMPLE 2.3.7***

Compute $\lim_{x \to 1} \frac{x^2 - 3x + 5}{x - 2}$ using the practical rules for calculating limits.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2 - 3x + 5}{x - 2} &= \frac{\lim_{x\to 1}(x^2 - 3x + 5)}{\lim_{x\to 1}(x - 2)} \\
&= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)} \\
&= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2} \\
&= \frac{1^2 - 3 \cdot 1 + 5}{1 - 2} = \frac{1 - 3 + 5}{-1} = -3
\end{aligned}$$

### *EXAMPLE 2.3.8*

Compute $\lim\limits_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$.

We cannot let $x = 1$, but we can factor out the hole.

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+3)}{(x-1)} = \lim_{x \to 1} (x + 3) = 4$$

### *EXAMPLE 2.3.11*

Compute $\lim\limits_{x \to 1} \frac{\sqrt{x+5} - 2}{x + 1}$.

$$\begin{aligned} \lim_{x \to -1} \frac{\sqrt{x+5} - 2}{x+1} &= \lim_{x \to -1} \frac{\sqrt{x+5} - 2}{x+1} \frac{\sqrt{x+5} + 2}{\sqrt{x+5} + 2} \\ &= \lim_{x \to -1} \frac{x + 5 - 4}{(x+1)(\sqrt{x+5} + 2)} \\ &= \lim_{x \to -1} \frac{x+1}{(x+1)(\sqrt{x+5} + 2)} \\ &= \lim_{x \to -1} \frac{1}{\sqrt{x+5} + 2} = \frac{1}{4} \end{aligned}$$

### ***One-sided limits***

- We say that **the limit from the left**, $\lim\limits_{x \to a^-} f(x) = L$, exists if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$.
- We say that **the limit from the right**, $\lim\limits_{x \to a^+} f(x) = L$, exists if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

### *EXAMPLE 2.3.13*

Discuss $\lim\limits_{x \to 0} \frac{x}{|x|}$, $\lim\limits_{x \to 0^-} \frac{x}{|x|}$, and $\lim\limits_{x \to 0^+} \frac{x}{|x|}$

The function is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and so $f(x) = -1$. Thus $\lim\limits_{x \to 0^-} \frac{x}{|x|} = \lim\limits_{x \to 0^-} -1 = -1$

and $\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} 1 = 1$. Because they aren't equal, the two-sided limit, $\lim_{x \to 0} \frac{x}{|x|}$, doesn't exist

***Infinity Limits***

- **Definition of a positive infinite limit**: $\lim_{x \to \infty} f(x) = L$ if $\varepsilon > 0$ if $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{R}$ s.t. $x > N$ then $|f(x) - L| < \varepsilon$
  - Same for **negative infinite limit**, except $x < N$

***Example: Negative Infinite Limit***

Find $\lim_{x \to -\infty} \frac{e^x}{x}$

$$\Rightarrow \quad \lim_{x \to -\infty} e^x \div \lim_{x \to -\infty} x \quad \Rightarrow \quad 0^+ \div -\infty \quad \Rightarrow \quad \lim_{x \to -\infty} \frac{e^x}{x} = 0^-$$

- $0^+$ means that it approaches 0 as the functions goes to infinity using positive numbers, and vice versa for $0^-$

***Special Limits to Memorize (TAKE AS GIVEN)***

1. $\lim_{x \to 0} \frac{sin\,(x)}{x} = 1$
2. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
3. $\lim_{x \to 0} \frac{cos\,(x) - 1}{x} = 0$

## 2.4 The Derivative Function

- **The Definition of a Derivative**: $\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
  - Note: the textbook uses $\Delta x$ instead of $h$
  - $\frac{dy}{dx}$ is called **Leibniz's notation** and comes from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
  - $f'(x)$ is called **prime notation**
- **Existence of a derivative**: $f$ is not ***differentiable*** at $x$ if
  1. There is a sharp corner, Ex. $\frac{d}{dx}|x|$ doesn't exist at $x = 0$
     a. Textbook example is that $\frac{d}{dx}x^{2/3}$ doesn't exist at $x = 0$
  2. The slope is vertical, Ex. $\frac{d}{dx}\sqrt[3]{x}$ doesn't exist at $x = 0$
  3. The function is not defined, i.e there is either a
     a. Jump, b. Asymptote c. or Hole

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### ***Example: Derivative of a Linear Function***

Find $f'(x)$ for $f(x) = 3x - 4$

$$\frac{d}{dx}f(x) = \lim_{h\to 0} \frac{[3(x+h) - 4] - [3x - 4]}{h} = \lim_{h\to 0} \frac{3x + 3h - 4 - 3x + 4}{h} = \lim_{h\to 0} \frac{3h}{h}$$

$\Rightarrow \lim_{h\to 0} 3 = 3$, thus $f'(x) = 3$

### ***Example: Derivative of a Quadratic Function***

Find $f'(x)$ for $f(x) = x^2$

$$\frac{d}{dx}f(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$\Rightarrow \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x + h) = 2$, thus $f'(x) = 2$

### ***Proof of $e^x$ from Lecture 2 (23 June, 2026)***

$\frac{d}{dx}e^x = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x * e^h - e^x}{h}$ break up term using an exponent rule

$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$ use special limit 2

$= e^x * 1 = e^x$

### ***Proof of $sin(x)$ from Lecture 2 (23 June, 2026)***

$\frac{d}{dx}sin(x) = \lim_{h\to 0} \frac{sin(x+h) - sin(x)}{h} = \lim_{h\to 0} \frac{sin(x)cos(h) + sin(h)cos(x) - sin(x)}{h}$ trig identity

$= \lim_{h\to 0} \frac{sin(x)[cos(h) - 1] + sin(h)cos(x)}{h} = \lim_{h\to 0} \frac{sin(x)[cos(h) - 1]}{h} + \lim_{h\to 0} \frac{sin(h)cos(x)}{h}$

$= sin(x) \lim_{h\to 0} \frac{[cos(h) - 1]}{h} + cos(x) \lim_{h\to 0} \frac{sin(h)}{h}$ use special limit 1 & 3

$= sin(x) * 0 + cos(x) * 1 = cos(x)$

## 2.5 Properties of Functions

- Graphs that pass the ***vertical line test*** are functional
- A function is **bounded** if its outputs stay within some finite interval. Mathematically, $\exists \{M \in \mathbb{R} \mid M > 0\}$ s.t. $|f(x)| \leq M \ \ \forall x$ in the domain
- A function is **continuous** if it is defined for every $x$ in its domain
    - Functions that aren't ***continuous*** can be **continuous at a points**
- If a function has a ***derivative*** $\forall x$, we say the function is **differentiable**

    - Similarly to the prior point, functions can be differentiable for some $x$ values even if not differentiable for all $x$ values
- **Intermediate Value Theorem**: if $f$ is continuous on the interval $[a, b]$ and $d$ is between $f(a)$ and $f(b)$, then $\exists$ some number $c$ in $[a, b]$ s.t. $f(c) = d$
    - A "common" use of this is **Bolzano's theorem**, if $f$ is continuous on the interval $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs (+/-), then $\exists$ a root/zero of $f$ in $[a, b]$